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THE PROBLEM OF THE STRESSED STATE OF AN ELASTIC CONE WEAKENED BY CRACKS†

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Using the method of discontinuous solutions, the problem of the stressed state of an elastic cone, weakened by cracks, is reduced to a system of one-dimensional integro-differential equations, specified on parts of the conical surfaces where the cracks are situated. There can be an arbitrary number of such surfaces and parts. The proposed scheme is realized using the example of the problem of the torsion of a cone, weakened by a semi-infinite conical crack, subjected to the action of an arbitrary load (including the application of a centre of rotation at the cone apex. An exact solution of this problem is obtained and a formula is given for the stress intensity factor. Since there is no solution in the literature of the problem of the stressed state of a cone without cracks due to the action of a centre of rotation, a solution is also given of this problem using the new integral transformation obtained here. It can also be used to solve problems of the stressed state of cones truncated along spherical surfaces. It follows from the problem of the stressed state of a cone, loaded with centre of rotation at the apex, which is solved here, that with type of loading the stress is everywhere equal to zero inside the cone, and hence a conical crack does not weaken the cone.

1. FORMULATION OF THE PROBLEM AND ITS REDUCTION TO A SYSTEM OF ONE-DIMENSIONAL INTEGRO-DIFFERENTIAL EQUATIONS

An elastic circular cone $(0 < r < \infty, -\pi < \varphi < \pi, 0 < \theta < \omega)$ with Poisson's ratio μ and shear modulus G, is loaded with an arbitrary static load on the surface $\theta = \omega$, i.e. $(\tau_{\theta r} \equiv \tau_r, \tau_{\theta\varphi} \equiv \tau_\varphi)$

$$\|\sigma_\theta(r,\omega,\varphi), \tau_r(r,\omega,\varphi), \tau_\varphi(r,\omega,\varphi)\| = \|f_1(r,\varphi), f_2(r,\varphi), f_3(r,\varphi)\| \tag{1.1}$$

including concentrated actions at the cone apex. The components of the displacement and stress fields in the cone due to these actions will be assumed known and we will denote them by a zero superscript. Inside the cone, along the surfaces $\theta = \omega_j$, $a_j \leqslant r \leqslant b_j$, $\omega_j < \omega$ $(j = 1, 2, \ldots, N)$ there are cracks, the edges of which $\theta = \omega_j - 0$ and $\theta = \omega_j + 0$ are assumed to be unloaded. It is required to determine the stress distribution in the elastic cone and to derive formulae for the stress intensity factor.

As previously [1], instead of the displacements u_r, u_θ, u_φ we introduce the functions

$$u(r,\theta,\varphi) = ru_r, \quad v(r,\theta,\varphi) = r\sin\theta\, u_\theta, \quad w(r,\theta,\varphi) = r\sin\theta\, u_\varphi \tag{1.2}$$

and their Fourier transformants

$$\|u_n(r,\theta), v_n(r,\theta), w_n(r,\theta)\| = \int_{-\pi}^{\pi} \frac{\|u(r,\theta,\varphi), v(r,\theta,\varphi), w(r,\theta,\varphi)\|}{2\pi\exp(in\varphi)} d\varphi \tag{1.3}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

and we take $u(r, \theta, \varphi)$, $\Theta(r, \theta, \varphi)$, $\Omega(r, \theta, \varphi)$ or $u_n(r, \theta)$, $\Theta_n(r, \theta)$, $\Omega_n(r, \theta)$ as the fundamental unknowns in terms of which the stresses are expressed by the formulae

$$\begin{aligned}
&(2G)^{-1}\sigma_{\theta n} = \mu_0\Theta_n - r^{-2}u_n' - r^{-2}\sin^{-2}\theta[v_n\sin\theta + in\, w_n\sin\theta] \\
&\mu_0 = (1-\mu)(1-2\mu)^{-1} \\
&\tau_{rn} = G\left[r\operatorname{cosec}\theta(r^{-2}v_n)' + r^{-2}u_n^{\bullet}\right] \\
&G^{-1}r\tau_{\varphi n} = \Omega_n + 2r^{-1}\sin^{-2}\theta[inv_n - w_n\cos\theta]
\end{aligned} \tag{1.4}$$

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The functions υ_n and w_n must be taken from formulae (3.6) or (3.7) from [1]. Here and below a prime denotes a derivative with respect to r (the first variable), while a dot denotes a derivative with respect to θ (the second variable).

Hence, the problem reduces to finding the functions $u_n(r, \theta)$, $\Theta_n(r, \theta)$, $\Omega_n(r, \theta)$. We will seek them in the form

$$\left\| u_n, \quad {}_n, \Omega_n \right\| = \left\| u_n^0 + \sum_{j=0}^{N} {}_j u_n, \Theta_n^0 + \sum_{j=0}^{N} {}_j\Theta_n, \Omega_n^0 + \sum_{j=0}^{N} {}_j\Omega_n \right\| \tag{1.5}$$

Here u_n^0, Θ_n^0, Ω_n^0 are known functions, which define the stress and displacement fields in the cone without friction (including that due to point forces at its apex), while for ${}_ju_n$, ${}_j\Theta_n$, ${}_j\Omega_n$ ($j = 1, 2, \dots, N$), which determine the discontinuous solution of Lame's equations for a conical defect (crack), situated on the surface $\theta = \omega_j$, we can use formulae (2.4)–(2.6) from [1], giving them preceding subscripts j, taking $\omega = \omega_j$ and replacing the integration sections $[0,R]$ by $[a_j,b_j]$. For the jumps in these functions and their derivatives normal to the defect formulae† (3.4) and (3.5) from [1] remain true, though in these equations we must put $\langle\tau_{rn}\rangle = \langle\sigma_{\theta n}\rangle = \langle\tau_{\varphi n}\rangle = 0$ and assume $\omega = \omega_j$. The functions ${}_j\upsilon_n(r,\theta)$ and ${}_jw_n(r,\theta)$, corresponding to these discontinuous solutions, are obtained from (3.6) and (3.7) from [1] with similar modifications.

When writing formulae (1.5) we used the idea of treating the boundary of the cone $\theta = \omega = \omega_0$ as an additional defect (a crack), which enables us to regard the terms ${}_0u_n$, ${}_0\Theta_n$, ${}_0\Omega_n$ as discontinuous solutions of Lamé's equations for this defect, where $a_0 = 0$ and $b_0 = \infty$. If we assume that the edge of this defect (the crack) is similarly loaded, the jumps in the stresses will be equal to zero, as for the specified cracks, while the jumps for the fundamental functions, which define the stress and displacement fields, will be determined by the same formulae as for the cracks, but in which we must put $j = 0$.

As can be seen, the problem will be solved if we obtain the transformants of the jumps in the displacements, i.e. $\langle{}_ju_n\rangle$, $\langle{}_j\upsilon_n\rangle$, $\langle{}_jw_n\rangle$ ($j = 1,2,\dots,N$). We will obtain the equations for determining them by realizing the condition that the crack edges should be stress-free, and also boundary conditions (1.1), written in terms of Fourier transformants, i.e. (δ_{jk} is the Kronecker delta)

$$\begin{gathered} \sigma_{\theta n}(r,\omega_l - 0) = \delta_{l0} f_{1n}(r), \quad \tau_{rn}(r,\omega_l - 0) = \delta_{l0} f_{2n}(r), \quad \tau_{\varphi n}(r,\omega_l - 0) = \delta_{l0} f_{3n}(r) \\ a_l \leqslant r \leqslant b_l, \quad l = 0,1,2,\dots N \end{gathered} \tag{1.6}$$

In order to avoid unnecessary complication, we will realize these conditions for the axisymmetrical case, i.e. when $n = 0$. In this case, realization of, for example, the third condition of (1.6) leads to the following system of integro-differential equations

$$\begin{aligned} &\langle {}_lw_0\rangle tg\omega - \int_{a_l}^{b_l} \left[\rho\langle {}_lw_0\rangle'' \Phi_0\left(\frac{r}{\rho},\theta,\omega_l\right)\sin\omega_l + 2\cos\omega_l \langle {}_lw_0\rangle \times \right. \\ &\left.\times \frac{\partial}{\partial\omega_l}\Phi_0\left(\frac{r}{\rho},\theta,\omega_l\right)\right]_{\theta=\omega_l} \frac{d\rho}{\rho} - \sum_{j=0}^{N}{}^{*} \sin\omega_l \int_{a_j}^{b_j} \left[\rho\langle {}_jw_0\rangle''\Phi_0\left(\frac{r}{\rho},\theta,\omega_j\right) + \right. \\ &\left. + 2\operatorname{ctg}\omega_j \langle {}_jw_0\rangle \frac{\partial}{\partial\omega_j}\Phi_0\left(\frac{r}{\rho},\theta,\omega_j\right)\right]_{\theta=\omega_l} \frac{d\rho}{\rho} + 2\operatorname{ctg}\omega_l \int_0^{\omega_l} \sum_{j=0}^{N} \int_{a_j}^{b_j} \left[\rho\langle {}_jw_0\rangle'' \times \right. \\ &\left. \times \Phi_0\left(\frac{r}{\rho},\tau,\omega_j\right) + 2\operatorname{ctg}\omega_j\langle {}_jw_0\rangle \frac{\partial}{\partial\omega_j}\Phi_0\left(\frac{r}{\rho},\tau,\omega_j\right)\right]\sin\tau d\tau = \\ &= \frac{r\sin\omega_l}{G}\left[\delta_{l0} f_{30}(r) - \tau_{\theta\varphi}^0(r,\omega_l)\right], \quad a_l \leqslant r \leqslant b_l, \quad l = 0,1,\dots N \end{aligned} \tag{1.7}$$

The asterisk on the summation sign denotes that the term with $j = 1$ is eliminated.

†In (2.4) from [1] there should be $\langle\dot{\Theta}_n\rangle$ and not $\langle\Theta_n\rangle$ in the first term on the right-hand side. This correction should be introduced into the last term in formula (2.5) as well; in addition, there should be $|n|$ instead of (n). In formula (3.3) of [1] there should be a plus sign after $\langle u_n\rangle$ (it was omitted).

We obtain a similar system after realizing the first and second conditions and (1.6), where it turns out to be compatible with the jumps $\langle_l v_0\rangle$ $\langle_j u_0\rangle$ and ($l = 0.1, \ldots, N$).

Hence, the problem splits into the problem of the twisting of a cone, which can be reduced to system of integro-differential equations (1.7) with respect to the jumps $\langle_j w_0\rangle$, and the problem of the axisymmetrical deformation of this cone, which reduces to a similar system of integro-differential equations in the jumps $\langle_l v_0\rangle$ and $\langle_j u_0\rangle$, where, on the right-hand side of this system, we have, instead of $\tau^0_{\theta\varphi}(r,\omega_t)$, the stresses $\sigma^0_\theta(r,\omega_t)$ and $\tau^0_{\theta\varphi}(r,\omega_t)$ resulting from the concentrated action at the cone apex. In this case there can only be a force acting along the cone axis.

A solution of this problem was obtained by Michell (see [2, 3]). In the case of twisting by a point force at the cone apex there can only be a centre of rotation, and the quantity $\tau^0_{\theta\varphi}(r,\omega_t)$ must be taken as the solution of the corresponding problem. Since there is no solution of this problem in the literature we will derive its solution. An attempt to proceed in the same way as Michell (see [2, 3]) did not lead to this objective, so we therefore used the idea [4] of replacing the cone in question by the frustum of a cone ($\varepsilon \leqslant r < \infty$), loading it on the spherical surface $r = \varepsilon$ with a suitable load and then letting ε approach zero. The realization of this idea required the solution of the problem of expanding an arbitrary function in the section $[\delta,\omega]$ in eigenfunctions of the regular Sturm–Liouville problem

$$\begin{aligned} &T''(\theta) + \operatorname{ctg}\theta T'(\theta) - [\lambda + \tfrac{1}{4} + m^2 \operatorname{cosec}^2\theta] T(\theta) = 0, \quad \delta < \theta < \omega \\ &[T'(\theta) - \operatorname{ctg}\theta T(\theta)]_{\theta=\delta,\omega} = 0 \end{aligned} \tag{1.8}$$

from which we must transfer to the case $\delta = 0$ required here (the irregular case) by taking the limit.

We will derive the solution of this problem especially as this expansion is necessary in order to solve more complex boundary-value problems for the frustum of a cone.

2. THE PROBLEM OF EXPANDING AN ARBITRARY CONTINUOUS FUNCTION IN ORTHOGONAL CONE FUNCTIONS

The solutions of the differential equation in (1.8) are the cone functions [6]

$$T(\theta) = P^m_{-1/2+i\sqrt{\lambda}}(\cos\theta), \quad Q^m_{-1/2+i\sqrt{\lambda}}(\cos\theta)$$

By making the replacement $y(\theta) = \sqrt{\sin\theta}T(\theta)$ we can reduce the Sturm–Liouville problem to the form [5]

$$\begin{aligned} &y''(\theta) - \{\lambda + q(\theta)\} y(\theta) = 0, \quad \delta < \theta < \omega \quad \left(q(\theta)\sin^2\theta = m^2 - \tfrac{1}{4}\right) \\ &y(\delta)\cos\alpha + y'(\delta)\sin\alpha = 0, \quad y(\omega)\cos\beta + y'(\omega)\sin\beta = 0 \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &\sin(\alpha,\beta) = \frac{1}{\Omega_\delta}\sin(\delta,\omega), \quad \cos(\alpha,\beta) = -\frac{3}{2\Omega_\omega}\cos(\delta,\omega) \\ &\Omega^2_\delta = \sin^2\delta + \frac{9}{4}\cos^2\delta \end{aligned} \tag{2.2}$$

Following the well-known scheme in [5], we proceed from the functions

$$\begin{aligned} &\varphi_0(\theta,\lambda) = \sqrt{\sin\theta}P^m_\nu(\cos\theta), \quad \chi_0(\theta,\lambda) = \sqrt{\sin\theta}Q^m_\nu(\cos\theta) \\ &\nu = -\tfrac{1}{2} + i\sqrt{\lambda} \end{aligned} \tag{2.3}$$

and we calculate their Wronksian, taking into account formula 3.4(25) from [6]. We obtain

$$\begin{aligned} &W(\varphi_0,\chi_0) = \omega_0 = -\Gamma_m(\nu), \quad \Gamma_m(\nu) = 2^{2m}(m+\nu)\Gamma[(m+\nu)/2]\times \\ &\times\ \Gamma[\tfrac{1}{2} + (m+\nu)/2]\{(\nu-m)\Gamma[(\nu-m)/2]\Gamma[\tfrac{1}{2} + (\nu-m)/2]\}^{-1} \end{aligned} \tag{2.4}$$

We further construct the functions

$$-\Omega_\delta^{(m)}\left\|\begin{matrix}\varphi(\theta,\lambda)\\ \chi(\theta,\lambda)\end{matrix}\right\| = \sqrt{\sin\theta}\left\|\begin{matrix}\sqrt{\sin\delta}F_\nu^m(\theta,\delta)\\ \sqrt{\sin\omega}F_\nu^m(\theta,\omega)\end{matrix}\right\|,\ \Omega_\delta^{(m)} = \Gamma_m(\nu)\Omega_\delta \tag{2.5}$$

$$F_\nu^m(\theta,\gamma) = P_\nu^m(\cos\theta)l_\gamma Q_\nu^m - Q_\nu^m(\cos\theta)l_\gamma P_\nu^m,\ l_\gamma = \sin\gamma\frac{d}{d\gamma} - \cos\gamma$$

and their derivatives

$$\Omega_\delta^{(m)}\left\|\begin{matrix}\varphi'(\theta,\lambda)\\ \chi'(\theta,\lambda)\end{matrix}\right\| = \sqrt{\sin\theta}\left\{\frac{d}{d\theta}\left\|\begin{matrix}\sqrt{\sin\delta}F_\nu^m(\theta,\delta)\\ \sqrt{\sin\omega}F_\nu^m(\theta,\omega)\end{matrix}\right\| + \frac{1}{2}\mathrm{ctg}\,\theta\left\|\begin{matrix}\sqrt{\sin\delta}F_\nu^m(\theta,\delta)\\ \sqrt{\sin\omega}F_\nu^m(\theta,\omega)\end{matrix}\right\|\right\}$$

The Wronksian of these functions is

$$W(\varphi,\chi) = \omega(\lambda) = -\sqrt{\sin\delta\sin\omega}\left[\Omega_\delta^{(m)}\Omega_\omega\right]^{-1}\Delta_\nu^{(m)}$$
$$\Delta_\nu^{(m)} = l_\delta P_\nu^m l_\omega Q_\nu^m - l_\delta Q_\nu^m l_\omega P_\nu^m \tag{2.6}$$

We thereby construct the function $\Phi(\theta, \lambda)$.

Suppose λ_k are the eigenvalues of problem (2.1). They must be found from the equation

$$\Delta_{\nu_k}^{(m)} \equiv l_\delta P_{\nu_k}^m l_\omega Q_{\nu_k}^m - l_\delta Q_{\nu_k}^m l_\omega P_{\nu_k}^m = 0,\ \nu_k = -1/2 + i\sqrt{\lambda_k} \tag{2.7}$$

from which we obtain the equality

$$l_\delta P_{\nu_k}^m\left[l_\omega P_{\nu_k}^m\right]^{-1} = l_\delta Q_{\nu_k}^m\left[l_\omega Q_{\nu_k}^m\right]^{-1} \tag{2.8}$$

According to this scheme [5], to obtain the required expansion it only remains to calculate the residue of the function $\Phi(\theta, \lambda)$ when $\lambda = \lambda_k$ and obtain K_k, which is equal to

$$K_k = \frac{\chi(\theta,\lambda_k)}{\varphi(\theta,\lambda_k)} = \frac{\sqrt{\sin\omega}\,\Omega_\delta l_\omega P_{\nu_k}^m}{\sqrt{\sin\delta}\,\Omega_\omega l_\delta P_{\nu_k}^m} \tag{2.9}$$

If we take into account the fact that

$$\frac{d\omega(\lambda)}{d\lambda} = \frac{d\omega}{d\nu}\frac{d\nu}{d\lambda} = -\frac{1}{2i\sqrt{\lambda}}\frac{d\omega}{d\nu} = -\frac{1}{2\nu+1}\frac{d\omega}{d\nu} \tag{2.10}$$

we obtain the required expansion [5] in the form

$$f(\theta) = \sum_{k=0}^{\infty}\frac{l_\delta Q_{\nu_k}^m\sqrt{\sin\theta}\,F_{\nu_k}^m(\theta,\omega)(2\nu_k+1)}{l_\omega Q_{\nu_k}^m\Gamma_m(\nu_k)\Delta_{mk}}\int_\delta^\omega F_{\nu_k}^m(\theta',\omega)\sqrt{\sin\theta'}\,f(\theta')d\theta'$$
$$\Delta_{mk} = \partial\Delta_\nu^m/\partial\nu\Big|_{\nu=\nu_k} \tag{2.11}$$

We take the limit as $\delta \to 0$ in this expansion. If in this case we take into account the asymptotic form of the Legendre functions in the neighbourhood of unity, then in Eq. (2.7) the second term will make the main contribution as $\delta \to 0$ and hence, in the limit, Eq. (2.7) becomes

$$l_\omega P_{\nu_k}^m = (\nu_k - m + 1)P_{\nu_k+1}^m(\cos\omega) - (\nu_k + 2)\cos\omega P_{\nu_k}^m(\cos\omega) = 0 \tag{2.12}$$

The first equality follows from 3.8(19) in [6]. On the same basis, after differentiating (2.7) with respect to ν (or ν_k), the main contribution will be made by a term of the form

$$-l_\delta Q_{\nu_k}\Delta_{mk}(\omega),\ \Delta_{mk}(\omega) = l_\omega\partial P_\nu^m/\partial\nu_\nu\Big|_{\nu=\nu_k} \tag{2.13}$$

and hence, as $\delta \to 0$ in (2.11), Δ_{mk} can be replaced by the quantity from (2.13). If we then use the fact that, as $\delta \to$, $F^m_{\nu_k}(\theta,\omega)$ becomes $l_\omega Q^m_{\nu_k} P^m_{\nu_k}(\cos\theta)$, expansion (2.11) becomes

$$f(\theta) = -\sum_{k=0}^{\infty} \sigma_{mk}(\omega)\sqrt{\sin\theta}\, P^m_{\nu_k}(\cos\theta)\int_0^{\omega} \sqrt{\sin\theta'}\, P^m_{\nu_k}(\cos\theta') f(\theta')d\theta'$$
$$\sigma_{mk}(\omega) = (2\nu_k + 1)l_\omega Q^m_{\nu_k}[\Delta_{mk}(\omega)\Gamma_m(\nu_k)]^{-1};\quad 0 \leq \theta \leq \omega \tag{2.14}$$

where the quantities ν_k must be found from transcendental equation (2.12).

We will give the asymptotic solution of this equation for large values of ν_k, using the asymptotic formula 3.9.1(2) from [6], retaining only the principal term. As a result, instead of (2.12) we will have

$$\text{tg}A = (m^2 + 2 + 3\nu_k/2)[m^2 + (\nu_k+1)^2]^{-1}\,\text{ctg}\,\omega$$
$$A = (\nu_k + \tfrac{1}{2})\omega - \pi/4 + m\pi/2 \tag{2.15}$$

We further show that

$$\text{tg}\,A = \begin{cases} \text{tg}[(\nu_k + \tfrac{1}{2})\omega - \pi/4], & m = 2j \\ -\text{ctg}[(\nu_k + \tfrac{1}{2})\omega - \pi/4], & m = 2j+1;\quad j = 0, 1\ldots \end{cases} \tag{2.16}$$

Consider the case when $m = 2j + 1$. We substitute expression (2.16) into (2.15) and, taking into account the fact that, as $\nu_k \to \infty$ the right-hand side of (2.15) tends to $a_k = 3(2\nu_k)^{-1}$ ctg ω, after obvious reduction using trigonometric functions we arrive at the equation

$$\text{tg}[(\nu_k + \tfrac{1}{2})\omega] = -(1+a_k)(1-a_k)^{-1}$$

Its right-hand side approaches minus unity as $\nu_k \to \infty$. Hence, we obtain the following asymptotic formula for the required ν_k

$$\nu_k = -\tfrac{1}{2} - (-\pi/4 \pm k\pi)\omega^{-1},\quad k = 0, 1, 2\ldots$$

Using the same transformations a similar formula can also be derived for $m = 2j$. The formula obtained only holds for large values of k. It is important to note that $\nu_0 = 1$ for any $m > 2$ and $m = 1$. This follows directly from (2.12), if we take into account the fact that [6]

$$P^m_n(z) = 0,\quad m > n \tag{2.17}$$

while for the case $m = 1$ one must additionally take into account formulae 8.813 from [7].

If we use formula (2.14) to expand the function $f(\theta) = (\sin\theta)^{-1/2}g(\theta)$, introducing for the function $g(\theta)$ its transformant

$$\int_0^{\infty} \sin\theta P^m_{\nu_k}(\cos\theta)g(\theta)d\theta = g_k \tag{2.18}$$

we can write formula (2.14) in the form of an inversion formula for this transformant

$$g(\theta) = -\sum_{k=0}^{\infty} \sigma_{mk}(\omega)g_k P^m_{\nu_k}(\cos\theta) \tag{2.19}$$

Consider the case when $\omega = \pi/2$. In this case the transformant equation (2.12) allows of the explicit solution $\nu_k = 2k - m$ (using formula 3.4 (20) from [6]), and formulae (2.18) and (2.19) become

$$\int_0^{\frac{1}{2}\pi} \sin\theta P^m_{2k-m}(\cos\theta)g(\theta)d\theta = g_k$$
$$g(\theta) = -\sum_{k=m}^{\infty} \sigma^*_{mk} P^m_{2k-m}(\cos\theta)g_k \tag{2.20}$$

$$\sigma^*_{mk} = \frac{(4k-2m+1)\Gamma(1+k-m)\Gamma(\tfrac{1}{2}+k-m)}{2^{2m}k!\Gamma(\tfrac{1}{2}+k)}$$

Formulae (2.18)–(2.20) enable us to solve not only the problem of the loading of a cone with a centre of rotation at the apex, but also more-complex boundary-value problems for the frustums of cones.

3. THE PROBLEM OF THE LOADING OF A CONE WITH A CENTRE OF ROTATION AT THE APEX

As indicated at the end of Section 1, we first solve the following problem. The frustum of the cone $\varepsilon \leqslant r < \infty$ is loaded along the spherical surface (the end of the cone) with a torsional load, i.e.

$$\tau_{r\varphi}|_{r=\varepsilon} = A\sin\theta, \quad A \equiv \text{const}, \quad 0<\theta<\omega \tag{3.1}$$

The torque, produced by this load, will be defined by the formula (compare with [8])

$$M = 2\pi\varepsilon^2 \int_0^\omega A\sin^3\theta d\theta = 4\pi\varepsilon^2 AA_\omega$$
$$A_\omega = \sin^2\frac{\omega}{2}\left[1-\frac{1}{3}(\cos^2\omega+\cos\omega+1)\right] \tag{3.2}$$

It is required to determine the displacements and stresses in the frustum of the cone.

The equation of the torsion of the frustum of the cone has the form [2, 3]

$$(r^2u'_\varphi)' + u^{\bullet\bullet}_\varphi + \operatorname{ctg}\theta u^{\bullet}_\varphi - \operatorname{cosec}^2\theta u_\varphi = 0, \quad \varepsilon \leqslant r < \infty, \quad 0<\theta<\omega \tag{3.3}$$

The stresses are expressed in terms of its solution by the formulae

$$\tau_{r\varphi} = Gr(r^{-1}u_\varphi)', \quad r\sin\theta\tau_{\theta\varphi} = Gl_\theta u_\varphi \tag{3.4}$$

The operator l_θ is defined in (2.5)

If we assume that the side surface of the cone is unloaded. the problem in question reduces to solving Eq. (3.3) with the boundary conditions

$$[r(r^{-1}u_\varphi)']_{r=\varepsilon} = AG^{-1}\sin\theta, \quad 0<\theta<\omega$$
$$l_\omega u_\varphi = 0, \quad \varepsilon \leqslant r < \infty \tag{3.5}$$

To solve boundary-value problem (3.3), (3.5) we will use integral transformation (2.18) and put $m = 1$ there. Using the standard scheme of the method of integral transformations, we obtain the solution of the problem in the form

$$u_\varphi(r,\theta) = -\frac{A\varepsilon^2}{Gr}\sum_{k=0}^{\infty}\frac{\gamma_k\sigma_{1k}(\omega)}{2+\nu_k}\left(\frac{\varepsilon}{r}\right)^{\nu_k} P^1_{\nu_k}(\cos\theta)$$
$$\gamma_k = \int_0^\omega \sin^2\theta P^1_{\nu_k}(\cos\theta)d\theta \tag{3.6}$$

We now take the limit as $\varepsilon \to 0$, simultaneously increasing the number A so that (cf. [4]) the torque (3.2) remains unchanged and equal to the torque at the centre of rotation M, i.e. we must have

$$4\pi A\varepsilon^2 = MA_\omega^{-1}, \quad \varepsilon \to 0 \tag{3.7}$$

If we take into account here the fact that $\nu_0 = 1$, $P^1_{\nu_0}(\cos\theta) = -\sin\theta$, $\nu_k > \nu_0$, $\gamma_0 = 2A\omega$, we obtain M

$$Gu_\varphi(r,\theta) = \frac{M}{6\pi r^2}\sigma_{10}(\omega)\sin\theta \tag{3.8}$$

In this case the stresses are

$$\tau_{r\varphi}(r,\theta)=-\frac{M}{2\pi r^3}\sigma_{10}(\omega)\sin\theta,\quad \tau_{\theta\varphi}(r,\theta)\equiv 0 \tag{3.9}$$

As can be seen, when a centre of rotation acts at the cone apex $0<r<\infty$, $|\varphi|\leqslant\pi$, $0<\theta<\omega$ the stresses vanish not only on the side surface ($\theta=\omega$), but also on any conical surface inside the cone, i.e. a cut in the cone along this surface does not weaken the cone.

In the special case when $\omega=\pi/2$, formulae (3.8) and (3.9) take the form

$$\mu_\varphi(r,\theta)=\frac{M}{\pi r^2}\sin\theta,\quad \tau_{r\varphi}(r,\theta)=-3\frac{M}{\pi r^3}\sin\theta,\quad \tau_{\theta\varphi}=0 \tag{3.10}$$

4. THE EXACT SOLUTION OF THE PROBLEM OF THE TORSION OF A CONE WEAKENED BY AN INFINITE CONICAL CUT

By the discussion in Sections 1 and 3, this problem, including the case when there is a centre of rotation at the apex, reduces to a system of two (l=0,1) equations (1.7), in which we must put

$$N=1,\quad a_0=0,\quad b_0=\infty,\quad a_1=R,\quad b_1=\infty,\quad \tau^0_{\theta\varphi}(r,\omega_j)=0,\quad j=0,1$$

and introduce the following notation for the unknown functions

$$\langle_j w_0(r,\omega_j)\rangle=\chi_j(r),\quad j=0,1;\quad \operatorname{supp}\chi_0(r)=[0,\infty],\quad \operatorname{supp}\chi_1(r)=[R,\infty] \tag{4.1}$$

We must then carry out transformations on the equations obtained, similar to those carried out previously [1] when obtaining Eq. (4.9) from [1] which are based on the two important relations†

$$\begin{gathered}-\int_{a_j}^{b_j}\langle_j w_0\rangle\Phi_0\left(\frac{r}{\rho},\ \omega_l,\ \omega_j\right)d\rho=r\frac{\partial}{\partial r}\int_{a_j}^{b_j}\frac{\chi_j(\rho)}{\rho}\Phi_0^0\left(\frac{r}{\rho},\ \omega_l,\ \omega_j\right)d\rho\\ r\frac{\partial}{\partial r}\Phi_k^1\left(\frac{r}{\rho}\right)=\Phi_k\left(\frac{r}{\rho}\right),\quad \Phi_k^1(t)=\begin{cases}k^{-1}t^k, & t<1\\ -(k+1)^{-1}t^{-k-1}, & t>1\end{cases},\quad k=1,2,\dots\end{gathered} \tag{4.2}$$

where

$$\begin{gathered}\Phi_0^0(t,\theta,\omega)=\sum_{k=0}^{\infty}\frac{P_k(\cos\theta)P_k(\cos\omega)}{2}\Phi_k^0(t)\\ \Phi_k^0(t)=\begin{cases}-(k+1)t^k, & t<1\\ kt^{-k-1}, & t>0\end{cases},\quad k=0,1,2,\dots\end{gathered} \tag{4.3}$$

The correctness of the first of relations (4.2) can be proved by integrating by parts, while the second can be checked directly. After carrying out these operations and making the replacements $r=xR$, $\rho=\xi R$ we arrive at the following system of equations

$$\begin{gathered}\operatorname{ctg}\omega_0\chi_0(Rx)+x\frac{d}{dx}\left[\int_0^\infty k_{00}\left(\frac{x}{\xi}\right)\chi_0(R\xi)\frac{d\xi}{\xi}+\int_1^\infty k_{01}\left(\frac{x}{\xi}\right)\chi_1(R\xi)\frac{d\xi}{\xi}\right]=f(x)\\ \operatorname{ctg}\omega_1\chi_1(Rx)+x\frac{d}{dx}\left[\int_1^\infty k_{11}\left(\frac{x}{\xi}\right)\chi_1(R\xi)\frac{d\xi}{\xi}+\int_0^\infty k_{10}\left(\frac{x}{\xi}\right)\chi_0(R\xi)\frac{d\xi}{\xi}\right]=0\\ 0\leqslant x<\infty,\ R\leqslant x<\infty\end{gathered} \tag{4.4}$$

†There are some printing errors in the relations in [1]. They are corrected here.

Here

$$f(x) = G^{-1}\sin(\omega_0)Rxf_{30}(Rx)$$

$$k_{jl} = \sin\omega_j k_{jl}^{(1)} - 2\operatorname{ctg}\omega_j k_{jl}^{(2)} - 2\operatorname{ctg}\omega_j k_{jl}^{(3)} + 4\operatorname{ctg}\omega_j k_{jl}^{(4)}; \quad j = 0, 1; \quad l = 0, 1$$

$$\left\| \begin{matrix} k_{jl}^{(1)} \\ k_{jl}^{(2)} \end{matrix} \right\| = \sum_{k=0}^{\infty} \frac{\Phi_k^0(t)}{2} \left\| \begin{matrix} P_k(\cos\omega_j) \\ J_{kj} \end{matrix} \right\| P_k(\cos\omega_l), \quad J_{kj} = \int_0^{\omega_j} P_k(\cos\tau)\sin\tau\, d\tau \tag{4.5}$$

$$\left\| \begin{matrix} k_{jl}^{(3)} \\ k_{jl}^{(4)} \end{matrix} \right\| = \sum_{k=0}^{\infty} \frac{\Phi_k^1(t)}{2} P_k^1(\cos\omega_l) \left\| \begin{matrix} P_k(\cos\omega_j) \\ J_{kl} \end{matrix} \right\|, \quad j = 0, 1; \quad l = 0, 1$$

In order to apply a Mellin transformation to these equations, we extend the second equation from (4.4) over the whole interval $(0, \infty)$. Since the left-hand side of this equation, by construction, is the function $B_\omega x\tau_{\theta\varphi}(Rx, \omega_1)$, $B_\omega = \sin\omega_1 RG^{-1}$, which us unknown in the interval $0 \leqslant x \leqslant 1$, by connecting it to the right-hand side of the equation in question we can carry out the necessary extension. After applying the Mellin transformation we obtain instead of (4.4)

$$\begin{aligned} &[\operatorname{ctg}\omega_0 - sK_{00}(s)]X^0(s) - sK_{01}(s)X^+(s) = F(s) \\ &[\operatorname{ctg}\omega_1 - sK_{11}(s)]X^+(s) - sK_{10}(s)X^0(s) = T^-(s) \end{aligned} \tag{4.6}$$

where

$$\left\| \begin{matrix} X^0(s) \\ F(s) \\ K_{jl}(s) \end{matrix} \right\| = \int_0^\infty \left\| \begin{matrix} \chi_0(Rx) \\ f(x) \\ k_{jl}(x) \end{matrix} \right\| x^{s-1}dx, \quad X^+(s) = \int_1^\infty \chi_1(Rx)x^{s-1}dx \tag{4.7}$$

$$T^-(s) = B_\omega \int_0^1 x\tau_{\theta\varphi}(Rx, \omega_1)x^{s-1}dx$$

The plus (minus) superscripts indicate functions that are analytical in the left (right) half-plane of the complex variable s. respectively. If we find $X^0(s)$ from the first equation of (4.6) and substitute the value obtained into the second equation, we arrive at the following Riemann boundary-value problem (9) (the functional Wiener–Hopf equation [10]), specified on the imaginary axis

$$K(s)X^+(s) = T^-(s) + H(s)F(s) \tag{4.8}$$

where

$$\begin{aligned} K(s) &= sK_{11}(s)\operatorname{ctg}\omega_1 - sK_{01}(s)H(s) \\ H(s) &= K_{10}(s)[s^{-1}\operatorname{ctg}\omega_0 - K_{00}(s)] \end{aligned} \tag{4.9}$$

In order to solve Riemann's problem (4.8), we need to factorize [9, 10] the function $K(s)$, and for this we need to know its behaviour as $s \to \infty$. A method of investigating the asymptotic form for $K_{jj}^m(s)$ ($j = 0, 1$: $m = 1, 2, 3, 4$) as $s \to \infty$ was proposed in [1]. In particular, it was shown that

$$2K_{jj}^{(1)}(s) = -\operatorname{ctg}\pi s + O(s^{-1}), \quad s \to \infty \tag{4.10}$$

However, this only holds when s tends to infinity, remaining on the imaginary axis (besides, this is sufficient for the results obtained in [1] to be correct). To refine the asymptotic form (4.10) we must sum the series

$$S^{\omega}(s)=\sum_{k=0}^{\infty}\left(\frac{1}{k+1-s}-\frac{1}{k+s}\right)\frac{\sin(2k+1)\omega}{2\pi}=S^{0}(s)-2S^{1}(s)$$
$$S^{0}(s)=\frac{1}{2}\pi\sum_{k=-\infty}^{\infty}\frac{\sin(2k+1)\omega}{k+s},\quad S^{1}(s)=\frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\omega}{k+s} \tag{4.11}$$

which is contained in formula (5.6) from [1]. The series $S^0(s)$, after making the replacement $2k+1 = m$, is summed using formula 5.4.3 (4) from [11] and takes the form

$$S^{0}(s)=\sin[(\pi-\omega)(2s-1)]\operatorname{cosec}\pi(2s-1) \tag{4.12}$$

while the series $S^1(s)$, by formula 5.4.6 (1) from [11], takes the form

$$S^{1}(s)=\frac{\sin\omega}{4\pi}\int_{0}^{\infty}\frac{e^{-st}(1+e^{t})}{\operatorname{ch}t-\cos 2\omega}dt \tag{4.13}$$

and according to the asymptotic expansion of the Laplace transformant [12] behaves at infinity as s^{-1}. Bearing relations (4.11)–(4.13) in mind, we obtain the refined asymptotic form

$$K_{jj}^{1}(s)=\tfrac{1}{2}\operatorname{ctg}\pi s+\operatorname{cosec}\pi\varsigma\sin(\pi-\omega_j)\varsigma+O(s^{-1}),\quad \varsigma=2s-1,\ s\to\infty \tag{4.14}$$

It can be shown that the additional term in (4.14) in fact decreases compared with (4.10), and exponentially when $s = i\sigma$, $\sigma\to\infty$.

Using the same methods as in [1], taking into account the transformation and (4.11), it can be shown that all $K_{jj}^{m}(s)$ ($j = 0, 1$: $m = 1, 2, 3, 4$) are decreasing functions as $s\to\infty$, but when $m = 1$ they may contain terms that are bounded at infinity, as in (4.12), i.e. we can write

$$K(s)=s[\tfrac{1}{2}\operatorname{ctg}\pi s+o(1)],\quad |s|\to\infty \tag{4.15}$$

while the symbol $o(1)$ contains terms which decrease exponentially at infinity if s approaches infinity, remaining on the imaginary axis, similar to the additional term in the asymptotic form (4.14) compared with (4.10).

Hence, the coefficient of Riemann's problem (4.8) has the same asymptotic form as in [1], and hence one can use the same method to factorize it. However, in this case it is more convenient to use a different approach, based on the fact that in the problem solved in [1] when $\omega = \pi/2$ (a half-space) we must factorize the function

$$-sK_{11}^{1}(s)\big|_{\omega_1=\pi/2}=-s\sum_{k=0}^{\infty}\frac{P_k^2(0)}{2}\int\Phi_k^0(t)t^{s-1}dt=$$
$$=-\frac{s(s-1)}{4}\sum_{j=0}^{\infty}\frac{(1/2)_j(1/2)_j}{(1)_j j!}\left[\frac{1}{j+s/2}+\frac{1}{j-(s-1)/2}\right]=L(s) \tag{4.16}$$

where

$$L(s)=L^{+}(s)L^{-}(s),\quad L^{+}(s)=\frac{\Gamma(\tfrac{3}{2}-s/2)}{\Gamma(1-s/2)},\quad L^{-}(s)=\frac{\Gamma(1+s/2)}{\Gamma(\tfrac{1}{2}+s/2)} \tag{4.17}$$

The last equality in (4.16) follows from formula 1.4(3) in [6].
Hence, in this case the factorization is carried out automatically, which was noted for the first time from other considerations in [13].

Using the properties of the gamma-function [6], we have the asymptotic form

$$2L(s)=s\operatorname{tg}\frac{\pi s}{2}\left[1+O\left(\frac{1}{s}\right)\right],\quad s\to\infty \tag{4.18}$$

This enables us to factorize the coefficient of Riemann's problem using the formula

$$K(s)=K^{+}(s)K^{-}(s),\quad K^{\pm}(s)=L^{\pm}(s)G^{\pm}(s) \tag{4.19}$$

where the function $G(s) = L^{-1}(s)K(s)$, which approaches unity as $s = i\sigma$ approaches $\pm\infty$ by virtue of the asymptotic forms (4.15) and (4.18), can be factorized using the well-known formula [9.14]

$$G^{\pm}(s)=\exp\left[\pm\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\ln[L^{-1}(t)K(t)]}{t-s}dt\right],\quad \operatorname{Re}s>0 \tag{4.20}$$

Carrying out standard operations using the factorization method [9, 15], bearing in mind, when using Liouville's theorem, the asymptotic form

$$L^{\mp}(s)=O(s^{1/2}),\quad G^{\mp}(s)=1+o(1),\quad s\to\infty \tag{4.21}$$

and also the mechanical meaning of the required functions, we obtain the solution of Riemann's problem (4.8) in the form

$$X^{+}(s)=[L^{+}(s)G^{+}(s)]^{-1}Q^{+}(s),\quad T^{-}(s)=L^{-}(s)G^{-}(s)Q^{-}(s) \tag{4.22}$$

Hence, from (4.17) we obtain a formula for the required stresses along the extension of the cut

$$B_{\omega}x\tau_{\theta\varphi}(Rx,\ \omega_1)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}Q^{-}(s)L^{-}(s)G^{-}(s)x^{-s}ds,\quad 0\leq x\leq 1,\quad \gamma>0 \tag{4.23}$$

The functions $Q^{\pm}(s)$ contained in (4.22) and (4.23) are found from the representation [9, 15]

$$Q(s)=H(s)F(s)[L^{-}(s)G^{-}(s)]^{-1}=Q^{+}(s)-Q^{-}(s) \tag{4.24}$$

and, for example, for the function $Q^{-}(s)$ we can obtain the formula [9, 10, 15]

$$Q^{-}(s)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{Q(t)}{t-s}dt,\quad Q^{-}(i\sigma)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{Q(i\tau)}{\tau-\sigma}d\tau \tag{4.25}$$

Having formula (4.23) we can obtain the stress intensity factor

$$K_{III}=\lim_{r\to R-0}\sqrt{2\pi(R-r)}\tau_{\theta\varphi}(r,\ \omega_1)=\sqrt{2\pi R}\lim_{x\to 1-0}\sqrt{1-x}\tau_{\theta\varphi}(Rx,\ \omega_1) \tag{4.26}$$

In order to take the limit we need to investigate the asymptotic form at infinity of the functions $Q^{-}(i\sigma)$, and to do this, using representation (4.25) and the results from [16], we must know the asymptotic form of $H^{-}(i\tau)$ as $\tau\to\infty$, for which, by (4.9), we need to use the asymptotic form for $K_{10}(s)$ taking transformation (4.11) into account. It has the same structure as (4.10) but only without the term ctg πs. Taking this into account we find that $H^{-}(i\tau) = O(\tau^{-1})$ $\tau\to\infty$.

If we now assume that the load applied to the cone is such that $\tau Q(i\tau)$ approaches zero and satisfies the Hölder condition there in the neighbourhood of ∞, by well-known results [16, p. 270] we will have

$$Q^{-}(i\sigma)=\frac{C}{\sigma}+o\left(\frac{1}{\sigma}\right),\quad \sigma\to\mp\infty,\quad C=\frac{1}{2\pi}\int_{-i\infty}^{i\infty}Q(t)dt \tag{4.27}$$

Taking this asymptotic form into account, the integral in (4.23) can be split into terms, as in [1], and we can separate the principal part, which carries the root singularity. In this case, this will be the integral

$$J(x)=\frac{C}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{L^{-}(s)}{s}x^{-s}ds=C\left[\frac{2\sqrt{\pi}}{\sqrt{1-x^2}(1+\sqrt{1-x^2})/2}+\frac{2}{\sqrt{\pi}}\right] \tag{4.28}$$

which can be evaluated using the theorem of residues using formula 2.8(6) from [6].

Taking the principal part of the integral in (4.23) in the form (4.28), and taking the limit of (4.26), we finally obtain

$$K_{III}=\frac{G}{2R}\frac{\sqrt{\pi}}{\sin\omega_1}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{H(s)F(s)}{L^{-1}(s)G^{-1}(s)}ds$$

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